TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 182, August 1973

ON THE ASYMPTOTIC BEHAVIOR OF NONLINEAR WAVE EQUATIONS

BY

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ABSTRACT. Positive energy solutions of the Cauchy problem for the equation $\Box u = m^2u + F(u)$ are considered. With $G(u) = \int_0^u F(s)\,ds$, it is proven that $G(u)$ must be nonnegative in order for uniform decay and the existence of asymptotic "free" solutions to hold. When $G(u)$ is nonnegative and satisfies a growth restriction at infinity, the kinetic and potential energies (with $m = 0$) are shown to be asymptotically equal. In case $F(u)$ has the form $|u|^{p-1}u$, scattering theory is shown to be impossible if $1 < p \le 1 + 2n^{-1}$ $(n \ge 2)$.

1. **Introduction.** Semilinear wave equations of the form

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u + F(u) = 0 \quad \left(\Delta = \text{Laplacian} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right) \tag{1}$$

provide simplified models for questions in relativistic quantum mechanics. Perhaps the most physically relevant example is the "meson" equation which is (1) in three space dimensions $(n = 3)$ with $F(u) = gu^3$ $(g > 0)$.

In §§2, 3, and 5, we consider the asymptotic behavior of certain solutions of the Cauchy problem for (1) with $x \in R^n$, $t \ge 0$, $m > 0$. Let $G(u) = \int_0^u F(s)\,ds$, and denote the energy norm $\|\cdot\|_e$ by

$$\|u(t)\|_e^2 = \frac{1}{2} \int_{R^n} [|u_t|^2 + |\nabla u|^2 + m^2|u|^2]\,dx. \tag{2}$$

In one form, scattering theory attempts to assert the existence of a solution $u_+(x, t)$ of the *free* equation ((1) with $F(u) \equiv 0$) to which a given solution $u(x, t)$ of (1) is asymptotic in energy norm as $t \to \infty$. The works of Strauss [12] and Morawetz and Strauss [8] show that, for $n = 3$ and for a nonnegative function $G(u)$ satisfying a growth condition at infinity, "fast enough" uniform decay (decay of $\sup_x |u|$) as $t \to \infty$ suffices to prove the existence of such a u_+.

Received by the editors September 5, 1972.

AMS (MOS) subject classifications (1970). Primary 35B40, 35L15.

Key words and phrases. Positive energy solutions, absence of uniform decay, nonexistence of scattering.

(1) This work constitutes part of the author's doctoral dissertation in the Division of Applied Mathematics at Brown University, June 1972, and was supported by an IBM Graduate Fellowship.

Roffman in [9] has given an example of a complex solution which does not decay uniformly to zero as $t \to \infty$, assuming that $\frac{1}{2}m^2u^2 + G(u) \geq 0$ and that $G(u_0) < 0$ for some u_0. His proof exploits a special invariant for complex solutions, the "charge", which vanishes identically in the real case. However, the absence of uniform decay does not, in itself, preclude the existence of such an asymptotic representation u_+.

We show first in §2 that in the complex case, under essentially Roffman's hypotheses, there exist solutions of (1) to which no finite energy free solution is asymptotic in energy norm as $t \to \infty$. In §3 we extend this result, and Roffman's theorem, to real solutions. It is shown that there exist real Cauchy data and a positive constant m_0, depending on this data, on n, and on the value of $G(u_0)$, for which the corresponding solution u of (1) is not asymptotic to any finite energy u_+ and, moreover, $\sup_x |u(x, t)| \not\to 0$ as $t \to \infty$, provided $0 < m \leq m_0$. Thus the condition that $G(u)$ be nonnegative cannot be violated if there is to be scattering theory for (1).

Along somewhat different lines, we combine results in [3] and [12] and obtain in §4 a simple proof showing that the kinetic and potential energies of certain nonlinear equations (with $m = 0$) are asymptotically equal.

In §5, we examine equation (1) with $F(u) = |u|^{p-1}u$ $(p > 1)$. When $n = 3$, it is known (cf. [8], [12]) that a scattering theory can be constructed for (1) if $3 \leq p < 5$. Moreover, Segal has obtained in [10] and [11] related asymptotic results provided $p > 2 + 2n^{-1}$. We show that if p is sufficiently small (more precisely, if $p \leq 1 + 2n^{-1}$) scattering theory is impossible for complex solutions of (1).

On Euclidean n-space R^n, L_p will denote the space of functions u on R^n whose pth powers are integrable, with the usual norm

$$\|u\|_p = \left(\int |u(x)|^p \, dx\right)^{1/p}; \qquad \|u\|_\infty = \text{ess sup } |u(x)|.$$

The positive integer n denotes the number of space dimensions. The energy norm has been defined by (2), where

$$\nabla u = \text{grad}_x \, u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$$

and, accordingly, Δu = Laplacian $u = \sum_{i=1}^n (\partial^2 u/\partial x_i^2)$.

An integral sign to which no domain is attached will be understood to be taken over all space. ω denotes a unit vector in R^n; $d\omega$ = element of surface measure on the unit sphere in R^n; ω_n = area of the unit sphere in R^n. The notation $v \in C_0^\infty(R^n)$ will mean, as usual, that v is infinitely differentiable and has compact support in R^n.

2. Complex solutions with $G(u)$ somewhere negative. Consider the Cauchy problem for the equation

(1) $$\partial^2 u/\partial t^2 - \Delta u + m^2 u + F(u) = 0 \qquad (x \in R^n,\ t > 0) \text{ where } m > 0.$$

Throughout §§2 and 3, we shall write

(2) $$G(u) = \int_0^u F(s)\,ds.$$

The *free equation* is (1) with $F(u) \equiv 0$, i.e. the Klein-Gordon equation. It is our goal to show that the condition "$G(u) \geq 0$ for all u" cannot be violated in order for uniform decay and the existence of asymptotic solutions to hold for (1). In what follows, we shall assume global existence and uniqueness for solutions of (1) with smooth data of compact support. In this regard, see [4], [5], [6], [11], [12], [13], and [14].

A multiplication of (1) by $\overline{\partial u/\partial t}$ and an integration over all space show that the *energy*

(3) $$E = \int_{R^n} [\tfrac{1}{2}|u_t|^2 + \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}m^2|u|^2 + G(u)]\,dx$$

is a constant independent of t. Thus if we assume that

(HI) $$\tfrac{1}{2}m^2 s^2 + G(s) \geq 0 \quad \text{for all real } s,$$

then solutions of (1) will have *positive* energy. Moreover, this energy will be finite. The "weakest" way to violate the condition that $G(u)$ be nonnegative is to assume that

(HII) $$G(u_0) < 0 \quad \text{for some } u_0.$$

Hypotheses (HI) and (HII) will be assumed to hold throughout §§2–3, and will not be repeated.

Roffman in [9] has shown that (1) has complex-valued solutions $u(x, t)$ such that $\sup_x |u(x, t)| \not\to 0$ as $t \to \infty$. He assumes (HI), (HII), that $\arg F(s) = \arg s$ for all s, and that $G(s) = o(|s|^2)$ as $s \to 0$. The assumption $\arg F(s) = \arg s$ implies that the function $Q(t) = m \operatorname{Im} \int \bar{u} u_t\,dx$ is a constant independent of time. We shall now prove that, under conditions very similar to those above, no finite energy u_+ can exist.

Theorem 1. *Consider equation* (1) *and assume that*

(i) $\arg F(s) = \arg s$ *for all* s;

(ii) $G(s) = O(|s|^{2+\delta})$ *as* $|s| \to 0$, *for arbitrary* $\delta > 0$; $G(s) = O(|s|^q)$ *as* $|s| \to \infty$, *where* q, $2 < q < \infty$, *is arbitrary for* $n = 1$ *or* 2, $q = 2n/(n-2)$ *for* $n \geq 3$.

Then (1) *has complex-valued solutions* $u(x, t)$ *with the property that: there does not exist any finite energy solution* $u_+(x, t)$ *of the free equation such that*

$$\|u(t) - u_+(t)\|_e \to 0 \quad \text{as } t \to \infty.$$

Proof. As noted above, in addition to the total energy E given by (3), (1) has another invariant $Q(t)$ defined by $Q(t) = m \operatorname{Im} \int \bar{u} u_t \, dx$. Aside from the constant factor m, $Q(t)$ is the "charge". Given a solution u of (1), suppose there is a finite energy solution u_+ of the free equation such that $\| u(t) - u_+(t) \|_e \to 0$ as $t \to \infty$. We shall first take the case $n \geq 3$, so that $q = 2n/(n-2)$. For any p, $2 < p \leq 2n/(n-2)$, we have

$$\|u(t)\|_p \leq \|u(t) - u_+(t)\|_p + \|u_+(t)\|_p.$$

Now u_+ is a finite energy solution of the free equation, that is,

$$E_+ = \text{const} = \tfrac{1}{2} \int [|\partial u_+/\partial t|^2 + |\nabla u_+|^2 + m^2 |u_+|^2] \, dx < \infty.$$

Using a result of Segal (see [10, Corollary 2, pp. 99–100]) we have that $\| u_+(t) \|_p \to 0$ as $t \to \infty$ for all such p. (This follows from the uniform decay of u_+ for compactly supported solutions, the density (in energy norm) of such solutions in all finite energy solutions, and the boundedness of $\| u_+(t) \|_2$.) For the other term we obtain by Hölder's inequality

$$\|u(t) - u_+(t)\|_p \leq \|u(t) - u_+(t)\|_2^{\theta} \, \|u(t) - u_+(t)\|_{2n/(n-2)}^{1-\theta}$$

where $1/p = \theta/2 + (1 - \theta)(n-2)/2n \quad (0 \leq \theta \leq 1)$.

Sobolev's theorem yields $\| u(t) - u_+(t) \|_{2n/(n-2)} \leq \text{const} \, \| u(t) - u_+(t) \|_e \to 0$ as $t \to \infty$; thus we have $\| u(t) - u_+(t) \|_p \to 0$ as $t \to \infty$. From this and the above it follows that $\| u(t) \|_p \to 0$ as $t \to \infty$, for all p, $2 < p \leq 2n/(n-2)$. Hypothesis (ii) gives an estimate of the form

$$|G(u)| \leq \text{const} \, (|u|^{2+\delta} + |u|^{2n/(n-2)})$$

for all u; hence $\int G(u) \, dx \to 0$ as $t \to \infty$, so that $\| u(t) \|_e^2 \to E$ as $t \to \infty$. Since $| \, \| u_+(t) \|_e - \| u(t) \|_e | \leq \| u(t) - u_+(t) \|_e \to 0$ as $t \to \infty$, and since u_+ is a free solution, we have that $\| u_+(t) \|_e^2 = E$ for all $t \geq 0$, and therefore $E_+ = E$. Now

$$\text{const} = |Q(t)| = \left| m \operatorname{Im} \int \bar{u} u_t \, dx \right| \leq \frac{m^2}{2} \|u(t)\|_2^2 + \frac{1}{2} \left\| \frac{\partial u}{\partial t}(t) \right\|_2^2$$

$$\leq \tfrac{1}{2} m^2 [\|u(t) - u_+(t)\|_2^2 + 2\|u_+(t)\|_2 \|u(t) - u_+(t)\|_2 + \|u_+(t)\|_2^2]$$

$$+ \frac{1}{2} \left[\left\| \frac{\partial u}{\partial t}(t) - \frac{\partial u_+}{\partial t}(t) \right\|_2^2 + 2 \left\| \frac{\partial u_+}{\partial t}(t) \right\|_2 \left\| \frac{\partial u}{\partial t}(t) - \frac{\partial u_+}{\partial t}(t) \right\|_2 + \left\| \frac{\partial u_+}{\partial t}(t) \right\|_2^2 \right]$$

$$\leq \|u_+(t)\|_e^2 + \|u(t) - u_+(t)\|_e^2 + \text{const} \, \|u(t) - u_+(t)\|_e = E + \epsilon(t)$$

where $\epsilon(t) \to 0$ as $t \to \infty$. Hence

$$|Q(t)| \leq E \quad \text{for all } t \geq 0. \tag{4}$$

Consider now the data $u(x, 0) = u_0 \zeta(x)$; $\partial u(x, 0)/\partial t = -imu_0 \zeta(x)$ where u_0 is as in (HII) and where $\zeta = \zeta(x)$ is a smooth nonnegative test function equal to 1 for $|x| \le R$, 0 for $|x| \ge R+1$, $|\nabla \zeta(x)| = O(1)$ for $R \le |x| \le R+1$. Then, as Roffman has shown, (HII) implies that $Q(0) = -m^2 u_0^2 \int \zeta^2(x)\,dx < -E$ for R sufficiently large, so that

$$|Q(0)| > E. \tag{5}$$

Since $Q(t)$ is constant, (4) and (5) are incompatible, the desired contradiction. This completes the proof for $n \ge 3$. When $n \le 2$, we have as above that $\|u_+(t)\|_p \to 0$ as $t \to \infty$ for *any* p, $2 < p < \infty$. From [2, pp. 24–27], we have the Sobolev inequalities

$$\|v(t)\|_p \le \text{const}\, \|v_x(t)\|_2^{\theta} \|v(t)\|_2^{1-\theta} \quad \text{for } n = 1,$$

where $1/p = (1-2\theta)/2$ $(0 \le \theta \le 1/2)$, and

$$\|v(t)\|_p \le \text{const}\, \|\nabla v(t)\|_2^{\theta} \|v(t)\|_2^{1-\theta} \quad \text{for } n = 2$$

where $1/p = (1-\theta)/2$ $(0 \le \theta < 1)$.

Applying these to the function $v = u - u_+$, we find that $\|u(t) - u_+(t)\|_p \to 0$ as $t \to \infty$ for all p, $2 < p < \infty$, which completes the proof.

Remark. The assertion (4) may also be derived by noting that $Q(t) = Q_+(t)$ for all $t \ge 0$, where $Q_+(t) = m \operatorname{Im} \int \bar{u}_+ (\partial u_+/\partial t)\,dx$. For

$$m^{-1}(Q(t) - Q_+(t)) = \operatorname{Im} \int \left(\bar{u} \frac{\partial u}{\partial t} - \bar{u}_+ \frac{\partial u_+}{\partial t} \right) dx$$

$$= \operatorname{Im} \int \left[\bar{u} \left(\frac{\partial u}{\partial t} - \frac{\partial u_+}{\partial t} \right) - \frac{\partial u_+}{\partial t} (\bar{u}_+ - \bar{u}) \right] dx.$$

The Schwarz inequality and the boundedness of $u(t)$ and $\partial u_+/\partial t$ in $L_2(R^n)$ then show that $|Q(t) - Q_+(t)| \to 0$ as $t \to \infty$; since $Q(t) - Q_+(t) = \text{const}$, the remark is verified.

3. **Real solutions with $G(u)$ somewhere negative.** We now wish to extend Roffman's theorem to real solutions. The assumptions below are precisely those of [9], with the exception of (i) of Theorem 1. Theorem 2 now shows that Roffman's result remains valid in the real case, provided m is sufficiently small.

Theorem 2. *Assume that $G(s) = o(s^2)$ as $s \to 0$. Then there exist real Cauchy data and a positive constant m_0, depending on this data, on n, and on the value of $G(u_0)$, such that if $0 < m \le m_0$, the corresponding solution $u(x, t)$ of* (1) *satisfies*

$$\sup_x |u(x,t)| \not\to 0 \quad \textit{as } t \to \infty.$$

Proof. Define $Q(t) = \int u u_t\,dx = \tfrac{1}{2}(d/dt) \int u^2\,dx$. Let $\zeta = \zeta(x)$ be a smooth, nonnegative test function equal to 1 for $|x| \le R$, 0 for $|x| \ge R+1$, $|\nabla \zeta(x)| =$

$O(1)$ for $R \le |x| \le R+1$. Let $u(x,t)$ be the smooth solution of (1) corresponding to the Cauchy data

$$u(x,0) = u_0\zeta(x), \quad u_t(x,0) = -mu_0\zeta(x)$$

where u_0 is as in (HII). We shall prove that under the conditions above, $\sup_x |u| \not\to 0$ as $t \to \infty$. To proceed by contradiction, suppose that $\sup_x |u| \to 0$ as $t \to \infty$, and define

$$h(t) = \int [\tfrac{1}{2}u_t^2 + \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{2}m^2u^2 + G(u) + me^{-t}uu_t]\,dx = E + me^{-t}Q(t).$$

Since u is assumed smooth, h is at least continuous. Now

$$\begin{aligned} h(0) &= \int [\tfrac{1}{2}m^2u_0^2\zeta^2 + \tfrac{1}{2}u_0^2|\nabla\zeta|^2 + \tfrac{1}{2}m^2u_0^2\zeta^2 + G(u_0\zeta) - m^2u_0^2\zeta^2]\,dx \\ &= \int (\tfrac{1}{2}u_0^2|\nabla\zeta|^2 + G(u_0\zeta))\,dx \\ &= \frac{\omega_n}{n}R^nG(u_0) + \int_R^{R+1}\rho^{n-1}\left[\int_{|\omega|=1}\{\tfrac{1}{2}u_0^2|\nabla\zeta(\rho\omega)|^2 + G(u_0\zeta(\rho\omega))\}\,d\omega\right]d\rho \\ &\le (\omega_n/n)R^nG(u_0) + (\omega_n/n)\cdot \text{const}\{(R+1)^n - R^n\} \end{aligned}$$

where the const is independent of both R and m. Therefore

$$h(0) \le (\omega_n R^n/n)[G(u_0) + \text{const}\{(1+1/R)^n - 1\}].$$

Since $G(u_0) < 0$ by (HII), we may choose R so large that $h(0) < 0$. Given this fixed value of R, we define (for $n \ge 3$, with similar expressions for $n = 1, 2$)

$$m_0 = ((2(n-2))^{1/2} - 1)/(R+1)$$

and assume that $0 < m \le m_0$. We shall now show that $h(t) > 0$ for sufficiently large t. First we find, using the order hypothesis on $G(u)$, that

$$\left|\int G(u)\,dx\right| \le \sup_x \left|\frac{G(u)}{u^2}\right| \int u^2\,dx = \delta(t)\int u^2\,dx$$

where $\delta(t) \to 0$ as $t \to \infty$. Thus for t sufficiently large, $E \ge (\tfrac{1}{2}m^2 - \delta(t))\int u^2 dx$ so that $u(t)$ is bounded in $L_2(R^n)$ and $\int G(u)dx \to 0$ as $t \to \infty$. By Schwarz' inequality, $|Q(t)|$ is bounded for all t; hence

$$h(t) \ge \tfrac{1}{2}\int (u_t^2 + |\nabla u|^2 + m^2u^2)\,dx - \delta_1(t) - \text{const}\ e^{-t}$$

where $\delta_1(t) \to 0$ as $t \to \infty$. Therefore for large t we have $h(t) \ge \tfrac{1}{2}E > 0$, as asserted.

Now, since h is continuous and is negative at the origin $t = 0$, the intermediate value theorem implies the existence of a point $T > 0$ such that $h(T) = 0$. Let t_0 be the first such T. We claim that $t_0 < 1/m$. For, by the definitions of $h(t)$ and t_0, we have

$$E + me^{-t}Q(t) \leq 0 \quad \text{for } 0 \leq t \leq t_0;$$

i.e.

$$E + \tfrac{1}{2}me^{-t}(d/dt)\|u(t)\|_2^2 \leq 0 \quad (0 \leq t \leq t_0).$$

We integrate this over the interval $[0, t_0]$ to get

$$Et_0 + \frac{m}{2}\int_0^{t_0} e^{-t}\frac{d}{dt}\|u(t)\|_2^2\,dt \leq 0$$

and then integrate by parts:

$$Et_0 + \frac{m}{2}\left[e^{-t_0}\|u(t_0)\|_2^2 - \|u(0)\|_2^2 + \int_0^{t_0} e^{-t}\|u(t)\|_2^2\,dt\right] \leq 0.$$

Neglecting the first and last positive terms in the bracket above, we obtain $Et_0 \leq \tfrac{1}{2}m\|u(0)\|_2^2$. Hypothesis (HI) is now used to estimate E at $t = 0$ from below:

$$E > \tfrac{1}{2}\int u_t^2(x, 0)\,dx = \tfrac{1}{2}m^2\|u(0)\|_2^2;$$

thus

$$\tfrac{1}{2}m^2\|u(0)\|_2^2 t_0 < Et_0 \leq \tfrac{1}{2}m\|u(0)\|_2^2$$

so that $t_0 < 1/m$ as claimed.

Using (HI) again, we find

$$\begin{aligned} 0 = h(t_0) &\geq \frac{1}{2}\int (u_t^2(x, t_0) + |\nabla u(x, t_0)|^2)\,dx \\ &\quad - me^{-t_0}\left|\int u(x, t_0)u_t(x, t_0)\,dx\right| \\ &> \frac{1}{2}\int |\nabla u(x, t_0)|^2\,dx - \frac{m^2}{2}\int u^2(x, t_0)\,dx \end{aligned}$$

since $e^{-t_0} < 1$; thus

$$\int |\nabla u(x, t_0)|^2\,dx < m^2\int u^2(x, t_0)\,dx. \tag{6}$$

Now u has compact support in x for each fixed t, so that the integral $\int u^2(x, t_0)\,dx$ extends only over the set

$$\{x: |x| \leq R + 1 + t_0\} \subset \{x: |x| \leq R + 1 + 1/m\}.$$

Hence, starting with (6) and using Poincaré's inequality (see [7, p. 95]), we obtain

$$\begin{aligned} \int |\nabla u(x, t_0)|^2\,dx &< m^2\int u^2(x, t_0)\,dx = m^2\int_{|x|\leq R+1+t_0} u^2(x, t_0)\,dx \\ &\leq m^2\int_{|x|\leq R+1+1/m} u^2(x, t_0)\,dx \leq \frac{m^2(R + 1 + 1/m)^2}{2(n-2)}\int |\nabla u(x, t_0)|^2\,dx. \end{aligned}$$

It follows that $2(n-2) < (R+1)^2m^2 + 2(R+1)m + 1$; i.e.

$$(R+1)^2m^2 + 2(R+1)m + (1 - 2(n-2)) > 0. \tag{7}$$

However, for $0 < m \leq m_0$, the quadratic expression in (7) is nonpositive, the desired contradiction. Thus we are done for $n \geq 3$. For $n \leq 2$, the proof proceeds exactly as above until inequality (6) is obtained. When $n = 2$ we use Sobolev's inequality (cf. [2, p. 25, Equation 9.11])

$$\| u(t_0) \|_2 \leq \tfrac{1}{2} \| (\partial u/\partial x_1)(t_0) \|_1^{1/2} \, \| (\partial u/\partial x_2)(t_0) \|_1^{1/2}$$

to get

$$\int |\nabla u(x, t_0)|^2 \, dx < m^2 \int u^2(x, t_0) \, dx \leq \frac{m^2}{4} \left\| \frac{\partial u}{\partial x_1}(t_0) \right\|_1 \left\| \frac{\partial u}{\partial x_2}(t_0) \right\|_1$$

$$\leq \frac{m^2}{8} \left[\left\| \frac{\partial u}{\partial x_1}(t_0) \right\|_1^2 + \left\| \frac{\partial u}{\partial x_2}(t_0) \right\|_1^2 \right] \leq \frac{m^2 \pi (R + 1 + t_0)^2}{8} \int |\nabla u(x, t_0)|^2 \, dx$$

$$< \frac{m^2 \pi (R + 1 + 1/m)^2}{8} \int |\nabla u(x, t_0)|^2 \, dx.$$

Thus we arrive at $(R+1)^2m^2 + 2(R+1)m + (1 - 8/\pi) > 0$ which leads to the same contradiction as above with a different value of m_0. When $n = 1$, we proceed as follows: First let $0 \leq x \leq R + 1 + t_0$. Then $u(x, t_0) = -\int_x^{R+1+t_0} u_x(\xi, t_0) \, d\xi$ so that

$$u^2(x, t_0) \leq (R + 1 + t_0 - x) \int_x^{R+1+t_0} u_x^2(\xi, t_0) \, d\xi$$

$$\leq (R + 1 + t_0 - x) \int_0^{R+1+t_0} u_x^2(\xi, t_0) \, d\xi.$$

Thus $\int_0^{R+1+t_0} u^2(x, t_0) \, dx \leq \tfrac{1}{2}(R + 1 + t_0)^2 \int_0^{R+1+t_0} u_x^2(\xi, t_0) \, d\xi$. When $-(R + 1 + t_0) \leq x \leq 0$, we write $u(x, t_0) = \int_{-(R+1+t_0)}^x u_x(\xi, t_0) \, d\xi$ and obtain in an analogous fashion

$$\int_{-(R+1+t_0)}^0 u^2(x, t_0) \, dx \leq \tfrac{1}{2}(R + 1 + t_0)^2 \int_{-(R+1+t_0)}^0 u_x^2(\xi, t_0) \, d\xi.$$

Therefore

$$\int u^2(x, t_0) \, dx = \int_{-(R+1+t_0)}^{R+1+t_0} u^2(x, t_0) \, dx$$

$$\leq \tfrac{1}{2}(R + 1 + t_0)^2 \left[\int_{-(R+1+t_0)}^0 u_x^2(\xi, t_0) \, d\xi + \int_0^{R+1+t_0} u_x^2(\xi, t_0) \, d\xi \right]$$

$$= \tfrac{1}{2}(R + 1 + t_0)^2 \int u_x^2(\xi, t_0) \, d\xi < \tfrac{1}{2}(R + 1 + 1/m)^2 \int u_x^2(\xi, t_0) \, d\xi.$$

Using this inequality in (6) we arrive at $(R+1)^2m^2 + 2(R+1)m - 1 > 0$ which again provides a contradiction for sufficiently small m.

Remarks on the Proof. It is thought that Theorem 2 is true for all values of m; we are unable to exploit the rapid decrease of the exponential factor to remove the restriction on the size of m. In fact, the exponential was inserted only to reduce the hypotheses of Theorem 2 to those of Roffman. With the exponential replaced by unity in the definition of $h(t)$, we obtain the same result under the somewhat more restrictive hypothesis

$$G(s) = o(|s|^{2+4/n}) \quad \text{as } s \to 0, \text{ for } n \geq 3.$$

Indeed, we need only show, under this condition and the premise $\sup_x |u| \to 0$ as $t \to \infty$, that $h(t)$ is nonnegative for sufficiently large t. Using Hölder's and Sobolev's inequalities, we have

$$\left|\int G(u)\,dx\right| \leq \sup_x \left|\frac{G(u)}{|u|^{2+4/n}}\right| \int |u|^2|u|^{4/n}\,dx$$

$$= \delta(t) \int |u|^2|u|^{4/n}\,dx \quad (\text{with } \delta(t) \to 0 \text{ as } t \to \infty)$$

$$\leq \delta(t)\left(\int |u|^{2n/(n-2)}\,dx\right)^{(n-2)/n}\left(\int u^2\,dx\right)^{2/n} \leq \delta_1(t)\int |\nabla u|^2\,dx$$

with $\delta_1(t) \to 0$ as $t \to \infty$. This estimate implies the nonnegativity of h for large t, and the proof then proceeds as above.

One further comment is in order. Notice that the hypotheses of Theorem 2 remain invariant under an elementary change of variables $t \to \alpha r$, $x \to \beta y$, α, β scalars. However, since the "size" of m depends on the value of $G(u_0)$, such an approach fails to enlarge the range of admissible values of m.

The proofs of Theorems 1 and 2 can now be combined to show that scattering theory is impossible for certain real solutions of (1), provided m is sufficiently small.

Theorem 3. *Assume that G satisfies hypothesis* (ii) *of Theorem* 1. *Then there exist real Cauchy data and a positive constant m_0, depending on the same quantities as in Theorem* 2, *such that the corresponding solution $u(x, t)$ of* (1) *is not asymptotic in energy norm to any finite energy solution u_+ of the free equation, provided $0 < m \leq m_0$.*

Proof. Given a solution $u(x, t)$ of (1), suppose that there exists a finite energy free solution $u_+(x, t)$ such that $\| u(t) - u_+(t) \|_e \to 0$ as $t \to \infty$. In Theorem 1 we have shown that this implies $\int G(u)\,dx \to 0$ as $t \to \infty$. Since $u(t) \to u_+(t)$ in energy norm, we have that $u(t) \to u_+(t)$ in $L_2(R^n)$, so that $u(t)$ is bounded over $L_2(R^n)$. Now $(\partial u/\partial t)(t)$ is bounded in $L_2(R^n)$ by (HI); hence the Schwarz inequality shows that $Q(t) = \int u u_t\,dx$ is bounded for all t. Thus, defining $h(t)$ as in Theorem 2 and using the same data as in that theorem, we have $h(0) < 0$, $h(t) > 0$ for sufficiently large t. We may then follow the proof of Theorem 2 verbatim to obtain the desired contradiction.

4. **Equipartition of energy.** Consider the Cauchy problem for the equation

(8) $$\partial^2 u/\partial t^2 - \Delta u + u^3 = 0 \quad (x \in R^3,\ t > 0)$$

with real C^∞ data of compact support. It is known that there exists a unique classical solution for all $t \geq 0$ (see for example [12]). The total energy E is constant:

$$E = \int (\tfrac{1}{2} u_t^2 + \tfrac{1}{2} |\nabla u|^2 + \tfrac{1}{4} u^4)\, dx < \infty.$$

Let us set

$$\text{Kinetic Energy} = Tu(t) = \tfrac{1}{2} \int u_t^2\, dx;$$

$$\text{Potential Energy} = Vu(t) = \tfrac{1}{2} \int |\nabla u|^2\, dx.$$

These energies are said to be (asymptotically) *equipartitioned* if $Tu(t) - Vu(t) \to 0$ as $t \to \infty$. The proof of this result for (8) is a simple consequence of results in [12] and corresponding statements for linear wave equations.

Let $u(x, t)$ be a C^2 solution of (8). Then we know from [12] that $\int u^4\, dx = O(t^{-2})$ as $t \to \infty$ and, moreover, that there exists a finite energy solution $u_+(x,t)$ of the free equation (the linear wave equation) such that $\| u(t) - u_+(t) \|_e \to 0$ as $t \to \infty$, where here $\| u(t) \|_e^2 = \tfrac{1}{2} \int (u_t^2 + |\nabla u|^2)\, dx$. It follows that $\| u_+(t) \|_e^2 = E < \infty$, for all $t \geq 0$. From [3], we may conclude that

$$\|(\partial u_+/\partial t)(t)\|_2 - \|\nabla u_+(t)\|_2 \to 0 \quad \text{as } t \to \infty.$$

Then

$$\left| \left\| \frac{\partial u}{\partial t}(t) \right\|_2 - \|\nabla u(t)\|_2 \right|$$

$$\leq \left| \left\| \frac{\partial u}{\partial t}(t) \right\|_2 - \left\| \frac{\partial u_+}{\partial t}(t) \right\|_2 \right| + \left| \left\| \frac{\partial u_+}{\partial t}(t) \right\|_2 - \|\nabla u_+(t)\|_2 \right|$$

$$+ \left| \|\nabla u_+(t)\|_2 - \|\nabla u(t)\|_2 \right|$$

$$\leq \left\| \frac{\partial u}{\partial t}(t) - \frac{\partial u_+}{\partial t}(t) \right\|_2 + \left| \left\| \frac{\partial u_+}{\partial t}(t) \right\|_2 - \|\nabla u_+(t)\|_2 \right| + \|\nabla u_+(t) - \nabla u(t)\|_2$$

$$\to 0 \quad \text{as } t \to \infty.$$

Hence

$$|Tu(t) - Vu(t)| = |\tfrac{1}{2}(\|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2)|$$

$$= \tfrac{1}{2} | \|u_t(t)\|_2 - \|\nabla u(t)\|_2 | (\|u_t(t)\|_2 + \|\nabla u(t)\|_2)$$

$$\leq \text{const}\ | \|u_t(t)\|_2 - \|\nabla u(t)\|_2 | \to 0 \quad \text{as } t \to \infty.$$

Thus the energies are asymptotically equal and, in addition, since $\int u^4\, dx \to 0$ as $t \to \infty$, we have $Tu(t) \to \tfrac{1}{2}E$, $Vu(t) \to \tfrac{1}{2}E$ as $t \to \infty$.

Clearly this result holds for any nonlinear function $F(u)$ (in place of u^3) which satisfies the restrictions set down in [12]. Moreover, the theorem is true in any dimension, provided the existence of such a finite energy u_+ can be established. Another proof of equipartition of energy for the linear wave equation may be found in [7, p. 106]. We also mention that Duffin in [1] has proved that, given data of compact support, the kinetic and potential energies of the linear wave equation actually become equal after a certain finite time, provided n is odd.

5. **The case** $F(u) = |u|^{p-1}u$. In contrast to the situation considered in §§2 and 3, power-law interactions of the form $F(u) = |u|^{p-1}u$ provide an everywhere nonnegative energy term $G(u)$. If $F(u)$ vanishes to a sufficiently high order at $u = 0$, many positive results in scattering theory have been obtained (cf. [8], [10], [11], and [12]). We examine the opposite case and show that if p is sufficiently small, then scattering theory is impossible.

Consider complex-valued solutions of the Cauchy problem for the equation

$$\partial^2 u/\partial t^2 - \Delta u + m^2 u + |u|^{p-1}u = 0 \qquad (x \in R^n,\ t > 0) \tag{9}$$

where $m > 0$, $p > 1$. We define as before the *charge* $Q(t)$ by $Q(t) = \text{Im} \int \bar{u} u_t \, dx$.

Theorem 4. *Let $u(x, t)$ be a C^2 solution of equation* (9) *with Cauchy data in $C_0^\infty(R^n)$ satisfying $Q(0) \neq 0$. Suppose that*

$$1 < p \leq 2 \quad \textit{if } n = 1; \qquad 1 < p \leq 1 + 2n^{-1} \quad \textit{if } n \geq 2.$$

Then there does not exist any free solution $u_+(x, t)$ in $C_0^\infty(R^n)$ such that $\|u(t) - u_+(t)\|_e \to 0$ as $t \to +\infty$.

Proof. Let $u(x, t)$ be a C^2 solution of (9) for which $Q(0) \neq 0$. Suppose that there exists a free solution $u_+ \in C_0^\infty(R^n)$ such that $\|u(t) - u_+(t)\|_e \to 0$ as $t \to \infty$. Define

$$Q_+(t) = \text{Im} \int \bar{u}_+ \frac{\partial u_+}{\partial t} \, dx.$$

We then have $Q(t) = Q_+(t)$ for all $t \geq 0$ (see the remark at the end of §2); thus in particular $Q_+(0) \neq 0$. Since $(\partial u_+/\partial t\,(t))$ is bounded in $L_2(R^n)$ by the energy equality, we obtain by Schwarz' inequality

$$0 < |Q_+(0)| = \left| \text{Im} \int \bar{u}_+ \frac{\partial u_+}{\partial t} \, dx \right| \leq \|u_+(t)\|_2 \left\| \frac{\partial u_+}{\partial t}(t) \right\|_2 \leq \text{const } \|u_+(t)\|_2.$$

Therefore there is a positive constant C_0 such that

$$\int |u_+(x, t)|^2 \, dx \geq C_0 \quad \text{for all } t \geq 0.$$

Let the data of the free solution u_+ be supported in the ball $|x| \leq k$. Then by the support property and Hölder's inequality, we have

$$0 < C_0 \leq \int |u_+(x,\ t)|^2\, dx = \int_{|x| \leq k+t} |u_+(x,\ t)|^2\, dx$$
$$\leq \left(\int_{|x| \leq k+t} |u_+(x,\ t)|^{p+1}\, dx \right)^{2/(p+1)} \left(\int_{|x| \leq k+t} 1\, dx \right)^{(p-1)/(p+1)}$$
$$\leq \text{const}\ t^{n((p-1)/(p+1))} \left(\int |u_+(x,\ t)|^{p+1}\, dx \right)^{2/(p+1)}$$

for $t \geq 1$, where p is as in equation (9). Thus there exists a positive constant C_1 such that

$$\int |u_+(x,\ t)|^{p+1}\, dx \geq c_1 t^{-n(p-1)/2} \quad \text{for all } t \geq 1.$$

Now set $H(t) = \int (u(\overline{\partial u_+/\partial t}) - u_+(\overline{\partial u/\partial t}))\, dx$. Using the differential equations, we find that

$$\dot{H}(t) = \int \left(\frac{\partial u}{\partial t} \frac{\overline{\partial u_+}}{\partial t} + u \frac{\overline{\partial^2 u_+}}{\partial t^2} - \frac{\partial u_+}{\partial t} \frac{\overline{\partial u}}{\partial t} - u_+ \frac{\overline{\partial^2 u}}{\partial t^2} \right) dx$$

$$= 2i \operatorname{Im} \int \frac{\partial u}{\partial t} \frac{\overline{\partial u_+}}{\partial t}\, dx + \int u(\overline{\Delta u_+} - m^2 \overline{u}_+)\, dx$$
$$- \int u_+(\overline{\Delta u} - m^2 \overline{u} - |u|^{p-1} \overline{u})\, dx$$

$$= 2i \operatorname{Im} \int \frac{\partial u}{\partial t} \frac{\overline{\partial u_+}}{\partial t}\, dx + 2m^2 i \operatorname{Im} \int u_+ \overline{u}\, dx$$
$$+ 2i \operatorname{Im} \int \nabla u_+ \cdot \overline{\nabla u}\, dx + \int u_+ |u|^{p-1} \overline{u}\, dx.$$

Therefore $\operatorname{Re} \dot{H}(t) = \operatorname{Re} \int u_+ |u|^{p-1} \overline{u}\, dx$. We have

$$\operatorname{Re} \dot{H}(t) = \operatorname{Re} \int u_+[|u|^{p-1}\overline{u} - |u_+|^{p-1}\overline{u}_+ + |u_+|^{p-1}\overline{u}_+]\, dx$$
$$= \int |u_+|^{p+1}\, dx + \operatorname{Re} \int u_+[|u|^{p-1}\overline{u} - |u_+|^{p-1}\overline{u}_+ + |u|^{p-1}\overline{u}_+ - |u|^{p-1}\overline{u}_+]\, dx$$
$$\geq c_1 t^{-n(p-1)/2} + \operatorname{Re} \int u_+ |u|^{p-1}(\overline{u} - \overline{u}_+)\, dx + \operatorname{Re} \int |u_+|^2(|u|^{p-1} - |u_+|^{p-1})\, dx$$
$$\geq c_1 t^{-n(p-1)/2} - \int |u_+||u|^{p-1}|u - u_+|\, dx - \int |u_+|^2 ||u|^{p-1} - |u_+|^{p-1}|\, dx$$
$$= c_1 t^{-n(p-1)/2} - I_1 - I_2.$$

Consider first the special case $p = 2$, $n = 1$ or 2. Recalling that our free solution u_+ satisfies $\| u_+(t) \|_\infty = O(t^{-n/2})$ as $t \to \infty$ (cf. [10]) we obtain

$$I_1 = \int |u_+||u||u - u_+|\, dx \leq \|u_+(t)\|_\infty \|u(t)\|_2 \|u(t) - u_+(t)\|_2$$
$$\leq \text{const}\ t^{-n/2} \|u(t) - u_+(t)\|_e = o(t^{-n/2})$$
$$= o(t^{-n(p-1)/2}) \quad \text{as } t \to \infty, \text{ when } p = 2.$$

We now take the general case $1 < p \leq 1 + 2n^{-1} < 2$ for $n \geq 3$, and $1 < p < 2$ for $n = 1, 2$. Then using Hölder's inequality, we get

$$\begin{aligned} I_1 &= \int |u_+||u|^{p-1}|u - u_+|\,dx = \int |u_+|^{p-1}|u_+|^{2-p}|u|^{p-1}|u - u_+|\,dx \\ &\leq \|u_+(t)\|_\infty^{p-1} \int |u_+|^{2-p}|u|^{p-1}|u - u_+|\,dx \\ &\leq \|u_+(t)\|_\infty^{p-1} \left(\int |u_+|^2\,dx\right)^{(2-p)/2} \\ &\quad \cdot \left(\int |u|^2\,dx\right)^{(p-1)/2} \left(\int |u - u_+|^2\,dx\right)^{1/2} \\ &\leq \text{const}\, \|u_+(t)\|_\infty^{p-1} \|u(t) - u_+(t)\|_e = o(t^{-n(p-1)/2}) \end{aligned}$$

as $t \to \infty$. In a similar fashion we have

$$\begin{aligned} I_2 &= \int |u_+|^2 \left||u|^{p-1} - |u_+|^{p-1}\right| dx = \int |u_+|^{p-1}|u_+|^{3-p}\left||u|^{p-1} - |u_+|^{p-1}\right| dx \\ &\leq \|u_+(t)\|_\infty^{p-1} \int |u_+|^{3-p}\left||u| - |u_+|\right|^{p-1} dx \end{aligned}$$

where we have used a simple computation showing that $\left||x|^{p-1} - |y|^{p-1}\right| \leq \left||x| - |y|\right|^{p-1}$ which is valid since $0 < p - 1 < 1$. Hence

$$\begin{aligned} I_2 &\leq \|u_+(t)\|_\infty^{p-1} \int |u_+|^{3-p}|u - u_+|^{p-1}\,dx \\ &\leq \text{const}\; t^{-n(p-1)/2} \left(\int |u_+|^2\,dx\right)^{(3-p)/2} \left(\int |u - u_+|^2\,dx\right)^{(p-1)/2} \\ &\leq \text{const}\; t^{-n(p-1)/2} \|u(t) - u_+(t)\|_e^{p-1} = o(t^{-n(p-1)/2}) \end{aligned}$$

as $t \to \infty$.

Thus both I_1 and I_2 satisfy the same estimate for sufficiently large t; it follows that there is a positive constant C_2 such that $\operatorname{Re} \dot{H}(t) \geq C_2 t^{-n(p-1)/2}$ for large enough t, say $t \geq T$. Hence

$$\operatorname{Re} H(2T) - \operatorname{Re} H(T) \geq C_2 \int_T^{2T} t^{-n(p-1)/2}\,dt \geq C_2 \int_T^{2T} t^{-1}\,dt = C_2 \log 2 > 0$$

since $p \leq 1 + 2n^{-1}$ by hypothesis. However, the Schwarz inequality gives

$$\begin{aligned} |H(t)| &= \left| \int \left(u \frac{\overline{\partial u_+}}{\partial t} - u_+ \frac{\overline{\partial u}}{\partial t} \right) dx \right| = \left| \int \left[u \left(\overline{\frac{\partial u_+}{\partial t} - \frac{\partial u}{\partial t}} \right) - \frac{\overline{\partial u}}{\partial t} (u_+ - u) \right] dx \right| \\ &\leq (\|u(t)\|_2 + \|\partial u(t)/\partial t\|_2)(\|u(t) - u_+(t)\|_e); \end{aligned}$$

thus $|H(t)| \to 0$ as $t \to \infty$ so that $|H(T)| + |H(2T)| \to 0$ as $T \to \infty$. A sufficiently large choice of T in inequality (10) above then yields the desired contradiction, and completes the proof.

We conjecture that the theorem remains valid in one dimension for any p, $1 < p \leq 3$, and moreover, for arbitrary n, that there does not exist any *finite energy* free solution u_+ asymptotically equal to u in energy norm.

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